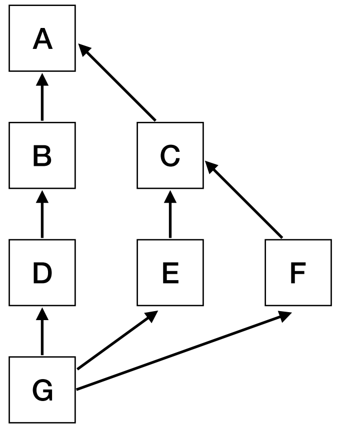
Exercise 1 : Specify Subtyping Relationship (10 points)

In this exercise we consider the system STLC extended with subtyping and a set of base types A, B, C, D, E, F, G. Subtyping between the *base types* is defined based on the following diagram:



In the diagram, the nodes represent types, and the arrows represent subtyping relationships between base types. For example, we have $B <: A$ and $E <: C$. The types in the system are defined as follows:

$$\begin{array}{lcll} \alpha & ::= & A \mid B \mid C \mid D \mid E \mid F \mid G & \text{base types} \\ S, T, U, L & ::= & \alpha \mid T \to T & \text{types} \end{array}$$

The subtyping rules are summarized below, which is standard:

$$\frac{\text{There is an arrow from } S \text{ to } T \text{ in the diagram}}{S \ <: \ T} \qquad \text{(S-BASE)}$$

$$T <: T \qquad \text{(S-REFL)}$$

$$\frac{S \ <: \ U \qquad U \ <: \ T}{S \ <: \ T} \qquad \text{(S-TRANS)}$$

$$\frac{T_1 <: S_1 \qquad S_2 <: T_2}{S_1 \to S_2 <: T_1 \to T_2} \qquad \text{(S-FUN)}$$

The *least upper bound* (LUB) and *greatest lower bound* (GLB) of types are specified as follows:

$$\begin{array}{lcl} LUB(T_1, T_2) = U & \iff & T_1 <: U \wedge T_2 <: U \wedge \forall U'.(T_1 <: U' \wedge T_2 <: U') \to U <: U' \\ GLB(T_1, T_2) = L & \iff & L <: T_1 \wedge L <: T_2 \wedge \forall L'.(L' <: T_1 \wedge L' <: T_2) \to L' <: L \end{array}$$

Part 1 (8 points). For each of the following pairs of types, compute LUB and GLB. If LUB or GLB does not exist, answer *None*.

1. B and C
   Solution. LUB: A, GLB: G

2. A and $A \to A$
   Solution. LUB: *None*, GLB: *None*

3. $D \to C$ and $A \to A$
   Solution. LUB: $D \to A$, GLB: $A \to C$

4. $G \to A$ and $(G \to A) \to B$
   *Solution.*LUB: *None*, GLB: *None*

5. $G \to D \to C$ and $G \to B \to A$
   *Solution.*LUB: $G \to D \to A$, GLB: $G \to B \to C$

Part 2 (2 points). Can we extend subtyping relationship to make LUBs and GLBs always exist for given examples? What changes to types and subtyping rules are needed?

Solution. We need to extend our system top and bottom types:

$$(\text{S-}\textsc{Bot})\ \bot\ <:\ T \qquad\qquad (\text{S-}\textsc{Top})\ T\ <:\ \top$$

Exercise 2 : Curry-Howard Correspondence (10 points)

The well-known *Curry-Howard correspondence* describes a mapping between type theory and logic: propositions correspond to types and proofs correspond to programs. This correspondence is usually formulated only for *intuitionistic logics* (IL), in which the *law of excluded middle* (LEM) or equivalently the *law of double negation* (DNE) does not hold. Concretely, the following propositions are not provable in IL, thus by the correspondence there exists no programs that prove them:

- LEM: $\forall P.P \vee \neg P$
- DNE: $\forall P.\neg\neg P \to P$

This problem is about proving that intuitionistic logic with the law of excluded middle is equivalent to intuitionistic logic with the law of double negation, that is $IL{+}LEM = IL{+}DNE$.

Curry-Howard: Negation. In intuitionistic logic, $\neg P$ is the same as $P \to \bot$, where $\bot$ means absurdity. So the type $\neg\neg P$ is interpreted as $(P \to \bot) \to \bot$. We assume absurdity $\bot$ corresponds to the type $\bot$ in types, which is not inhabited. The following program *explode* is provided:

$$explode : \forall P.\bot \to P$$

The program *explode* has the type $\forall P.\bot \to P$. Logically, it says that from absurdity any proposition can be derived, which corresponds to a well-known principle in logic.

Curry-Howard: Universal quantification and System F. A second-order proposition of form $\forall P.T$ corresponds to a type in System F and can be proved by a System F term. For example, $\forall P.P \to P$ is proved by the program $\Lambda P.\lambda x : P.x$.

Task. Please prove the following propositions. The last two prove that $IL{+}LEM = IL{+}DNE$:

(1) $\forall P.P \to \neg\neg P$

prog1 $= \Lambda P.\lambda x : P \lambda f : P \to \bot.f x$

(2) $\forall P.\neg\neg(P \vee \neg P)$

prog2 $= \Lambda P.\lambda f : (P + P \to \bot) \to \bot.$
$\quad let\ a : P \to \bot = \lambda x : P.f(inl\ x)\ in$
$\quad let\ b : (P \to \bot) \to \bot = \lambda x : P \to \bot.f(inr\ x)\ in$
$\quad b\ a$

(3) $(\forall P.P \vee \neg P) \to (\forall Q.\neg\neg Q \to Q)$

prog3 $= \lambda x : \forall P.(P + P \to \bot).\Lambda Q.\lambda f : (Q \to \bot) \to \bot.$
$\quad case(x\ [Q])\ of$
$\quad\quad inl\ q \Rightarrow q$
$\quad\quad inr\ nq \Rightarrow explode\ [Q](f\ nq)$

(4) $(\forall P.\neg\neg P \to P) \to (\forall Q.Q \vee \neg Q)$

prog4 $= \lambda x : (\forall P.((P \to \bot) \to \bot) \to P).\Lambda Q.x\ [Q + Q \to \bot](M\ [Q])$
where M is the term in (2).

Exercise 3 : Transitivity of Algorithmic Subtyping in $F_{<:}$ (10 points)

In this problem, we study algorithmic subtyping in System $F_{<:}$. System $F_{<:}$ is an extension of System F with subtyping of types and bounds on type variables. The types in System $F_{<:}$ are defined as follows:

$$T ::= \mathit{Top} \mid X \mid T \to T \mid \forall X <: T.T$$

One approach to formulate subtyping in $F_{<:}$ is algorithmic subtyping. The subtyping rules are given as follows:

$$\Gamma \vdash T <: \mathit{Top} \qquad \text{(S-Top)}$$

$$\Gamma \vdash X <: X \qquad \text{(S-TVar-Refl)}$$

$$\frac{X <: T \in \Gamma \qquad \Gamma \vdash T <: U}{\Gamma \vdash X <: U} \qquad \text{(S-TVar-Trans)}$$

$$\frac{\Gamma \vdash T_1 <: S_1 \qquad \Gamma \vdash S_2 <: T_2}{\Gamma \vdash S_1 \to S_2 <: T_1 \to T_2} \qquad \text{(S-Fun)}$$

$$\frac{\Gamma, X <: U \vdash S <: T}{\Gamma \vdash \forall X <: U.S \;<:\; \forall X <: U.T} \qquad \text{(S-All)}$$

For this problem, we may assume the typing environment Γ to be just a list of type bounds:

$$\Gamma ::= \emptyset \mid \Gamma, X <: T$$

The typing environment Γ is used in the rule S-TVar-Trans, and it is augmented in the rule S-All. For simplicity, in the rule S-All, we require the bound of two universal types to be the same type U.

Please prove the following theorem in the subtyping system.

Theorem 1 (Transitivity). *If* $\Gamma \vdash S <: U$ *and* $\Gamma \vdash U <: T$*, then* $\Gamma \vdash S <: T$.

Proof. Induction on the type derivation $\Gamma \vdash S <: U$. There are five cases.

- S-Top. We have $U = \mathit{Top}$. Invert $\Gamma \vdash \mathit{Top} <: T$, we know $T = \mathit{Top}$. Apply S-Top.
- S-Tvar-Refl. We have $S = U = X$, trivial.
- T-Tvar-Trans. We have $S = X$, $X <: Q \in \Gamma$ and $Q <: U$. By IH, we have $Q <: T$. Now apply T-Tvar-Trans.
- T-Fun. We have $S = S_1 \to S_2$, $U = U_1 \to U_2$, $U_1 <: S_1$ and $S_2 <: U_2$.

  Invert $\Gamma \vdash U_1 \to U_2 <: T$, either $T = \mathit{Top}$, we apply S-Top. Or we have $T = T_1 \to T_2$ and $T_1 <: U_1, U_2 <: T_2$. By IH, we have $T_1 <: S_1$ and $S_2 <: T_2$. By S-Fun, we have $\Gamma \vdash S <: T$.
- T-All. We have $U = \forall X <: Q.U_1$, $S = \forall X <: Q.S_1$ and $\Gamma, X <: Q \vdash S_1 <: U_1$.

  Invert $\Gamma \vdash \forall X <: Q.U_1 < T$, either $T = \mathit{Top}$, we apply S-Top. Or we have $T = \forall X.T_1$ and $\Gamma, X <: Q \vdash U_1 <: T_1$. By IH, we have $S_1 <: T_1$. Now apply S-All.

$\square$

For reference: **Simply Typed Lambda Calculus**

The complete reference of the simply typed lambda calculus is:

$$
\begin{array}{lcll}
t & ::= & & \textbf{terms}: \\
 & | & x & \textit{variable} \\
 & | & \lambda\, x\!:\!\mathtt{T}.\ t & \textit{abstraction} \\
 & | & t\ t & \textit{application} \\
 \\
v & ::= & & \textbf{values}: \\
 & | & \lambda\, x\!:\!\mathtt{T}.\ t & \textit{abstraction} - \textit{value} \\
 \\
\mathtt{T} & ::= & & \textbf{types}: \\
 & | & \mathtt{T} \to \mathtt{T} & \textit{type of functions (right assoc.)}
\end{array}
$$

Evaluation rules:

$$\frac{t_1 \longrightarrow t_1'}{t_1\ t_2 \longrightarrow t_1'\ t_2} \qquad \text{(E-App1)}$$

$$\frac{t_2 \longrightarrow t_2'}{v_1\ t_2 \longrightarrow v_1\ t_2'} \qquad \text{(E-App2)}$$

$$(\lambda\ x\colon \mathtt{T}_1.\ t_1)\ v_2 \longrightarrow [x \to v_2]\ t_1 \qquad \text{(E-AppAbs)}$$

Typing rules:

$$\frac{x : \mathtt{T} \in \Gamma}{\Gamma \vdash x : \mathtt{T}} \qquad \text{(T-Var)}$$

$$\frac{\Gamma,\ x : \mathtt{T}_1 \vdash t_2 : \mathtt{T}_2}{\Gamma \vdash (\lambda\ x\colon \mathtt{T}_1.\ t_2) : \mathtt{T}_1 \to \mathtt{T}_2} \qquad \text{(T-Abs)}$$

$$\frac{\Gamma \vdash t_1 : \mathtt{T}_1 \to \mathtt{T}_2 \quad \Gamma \vdash t_2 : \mathtt{T}_1}{\Gamma \vdash t_1\ t_2 : \mathtt{T}_2} \qquad \text{(T-App)}$$