Theory of Types and Programming Languages Fall 2022

Week 2

Reading Material

We're starting with Chapter 3 of the textbook.

(Chapter 2 contains some mathematical preliminaries which we assume you are familiar with.)

Where we're going

Going Meta...

The functional programming style (OCaml, Scala, Lisp) is based on treating *programs as data*

- *i.e.*, writing functions that manipulate other functions

Everything in this course is based on treating *programs as mathematical objects*

— *i.e.*, build mathematical theories whose basic objects of study are programs (and whole programming languages)

Jargon: We study the metatheory of programming languages

Warning!

The material in the next couple of lectures is more "slippery" than it may first appear

"I believe it when I hear it" not a sufficient test of understanding

Much better test: "I can explain it so that someone else believes it"

"You never really misunderstand something until you try to teach it..." — Anon.

Basics of Induction (Review)

Induction

Principle of *ordinary induction* on natural numbers:

Suppose that P is a predicate on the natural numbers. Then:

If P(0)and, for all *i*, P(i) implies P(i+1), then P(n) holds for all *n*.

Example

Theorem: $2^0 + 2^1 + ... + 2^n = 2^{n+1} - 1$, for every *n*.

Example

- Theorem: $2^0 + 2^1 + ... + 2^n = 2^{n+1} 1$, for every *n*.
- Proof: Let P(i) be " $2^0 + 2^1 + ... + 2^i = 2^{i+1} 1$." Show P(0): $2^0 = 1 = 2^1 - 1$

Show that P(i) implies P(i+1):

$$2^{0} + 2^{1} + \dots + 2^{i+1} = (2^{0} + 2^{1} + \dots + 2^{i}) + 2^{i+1}$$

= $(2^{i+1} - 1) + 2^{i+1}$ by IH
= $2 \cdot (2^{i+1}) - 1$
= $2^{i+2} - 1$

The result (P(n) for all n) follows by the principle of (ordinary) induction.

Shorthand form

Theorem: $2^0 + 2^1 + ... + 2^n = 2^{n+1} - 1$, for every *n*.

Proof: By induction on *n*.

Base case
$$(n = 0)$$
:

$$2^0 = 1 = 2^1 - 1$$

Inductive case
$$(n = i + 1)$$
:
 $2^{0} + 2^{1} + \dots + 2^{i+1} = (2^{0} + 2^{1} + \dots + 2^{i}) + 2^{i+1}$
 $= (2^{i+1} - 1) + 2^{i+1}$ IH
 $= 2 \cdot (2^{i+1}) - 1$
 $= 2^{i+2} - 1$

Complete Induction

Principle of *complete induction* on natural numbers:

Suppose that P is a predicate on the natural numbers. Then:

If, for each natural number n, given P(i) for all i < n we can show P(n), then P(n) holds for all n.

Complete versus ordinary induction

Ordinary and complete induction are *interderivable* — *assuming one, can prove the other*

Thus, the choice of which to use is purely a question of style

We'll see some other (equivalent) styles as we go along



Simple Arithmetic Expressions

"BNF" grammar for very simple language of arithmetic expressions:

terms
constant true
constant false
conditional
constant zero
successor
predecessor
zero test

Terminology:

t

t here is a metavariable

Abstract vs. concrete syntax

Q: Does this grammar define a set of *character strings*, a set of *token lists*, or a set of *abstract syntax trees*?

Abstract vs. concrete syntax

Q: Does this grammar define a set of *character strings*, a set of *token lists*, or a set of *abstract syntax trees*?

A: In a sense, all three. But we are primarily interested, here, in abstract syntax trees.

For this reason, grammars like the one on the previous slide are sometimes called *abstract grammars*. An abstract grammar *defines* a set of abstract syntax trees and *suggests* a mapping from character strings to trees.

We then *write* terms as linear character strings rather than trees simply for convenience. If there is any potential confusion about what tree is intended, we use parentheses to disambiguate. Q: So, are

```
succ 0
succ (0)
(((succ (((((0))))))))
```

"the same term"?

What about

```
succ 0
pred (succ (succ 0))
```

?

A more explicit form of the definition

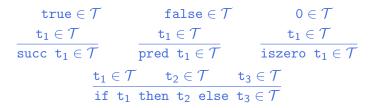
The set \mathcal{T} of *terms* is the smallest set such that

- 1. {true, false, 0} $\subseteq \mathcal{T}$;
- 2. if $t_1 \in \mathcal{T}$, then {succ t_1 , pred t_1 , iszero t_1 } $\subseteq \mathcal{T}$;
- 3. if $\mathtt{t}_1\in\mathcal{T}$, $\mathtt{t}_2\in\mathcal{T}$, and $\mathtt{t}_3\in\mathcal{T}$, then

if t_1 then t_2 else $t_3 \in \mathcal{T}$.

Inference rules

An alternate notation for the same definition:



Note that "the smallest set closed under..." is implied (but often not stated explicitly).

Terminology:

- axiom vs. rule
- concrete rule vs. rule scheme

Terms, concretely

Define an infinite sequence of sets, S_0 , S_1 , S_2 , ..., as follows:

$$\begin{array}{rcl} \mathcal{S}_0 &=& \emptyset \\ \mathcal{S}_{i+1} &=& \{\texttt{true}, \texttt{false}, 0\} \\ && \cup & \{\texttt{succ } \texttt{t}_1, \texttt{pred } \texttt{t}_1, \texttt{iszero } \texttt{t}_1 \mid \texttt{t}_1 \in \mathcal{S}_i\} \\ && \cup & \{\texttt{if } \texttt{t}_1 \texttt{ then } \texttt{t}_2 \texttt{ else } \texttt{t}_3 \mid \texttt{t}_1, \texttt{t}_2, \texttt{t}_3 \in \mathcal{S}_i\} \end{array}$$

Now let

 $S = \bigcup_i S_i$

Comparing the definitions

We have seen two different presentations of terms:

- 1. as the *smallest* set that is *closed* under certain rules (\mathcal{T})
 - explicit inductive definition
 - BNF shorthand
 - inference rule shorthand

2. as the limit (S) of a series of sets (of larger and larger terms)

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 - explicit inductive definition
 - BNF shorthand
 - inference rule shorthand
- 2. as the limit (S) of a series of sets (of larger and larger terms)

These presentations are equivalent.

Induction on Syntax

Why two definitions?

The two ways of defining the set of terms are both useful:

- 1. the definition of terms as the smallest set with a certain closure property is compact and easy to read
- 2. the definition of the set of terms as the limit of a sequence gives us an *induction principle* for proving things about terms...

Definition: The *depth* of term t is the smallest *i* such that $t \in S_i$.

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From the definition of S, it is clear that, if a term t is in S_i , then all of its immediate subterms must be in S_{i-1} , i.e., they must have strictly smaller depths.

This observation justifies the *principle of induction on terms*. Let P be a predicate on terms.

If, for each term s,
 given P(r) for all immediate subterms r of s
 we can show P(s),
then P(t) holds for all t.

Inductive Function Definitions

The set of constants appearing in a term t, written Consts(t), is defined as follows:

Simple, right?

 $\cup Consts(t_3)$

First question:

Normally, a "definition" just assigns a convenient name to a previously-known thing. But here, the "thing" on the right-hand side involves the very name that we are "defining"!

So in what sense is this a definition??

Second question:

Suppose we had written this instead...

The set of constants appearing in a term t, written BadConsts(t), is defined as follows:

 $BadConsts(true) = \{true\}$ BadConsts(false) = {false} = {0} BadConsts(0) $BadConsts(0) = \{\}$ $BadConsts(succ t_1) = BadConsts(t_1)$ $BadConsts(pred t_1) = BadConsts(t_1)$ $BadConsts(iszero t_1) = BadConsts(iszero (iszero t_1))$

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What is the essential difference between these two definitions? How do we tell the difference between well-formed inductive definitions and ill-formed ones? What, exactly, does a well-formed inductive definition mean?

What is a function?

Recall that a *function* f from A (its domain) to B (its co-domain) can be viewed as a two-place *relation* (called the "graph" of the function) with certain properties:

It is total: Every element of its domain occurs at least once in its graph. More precisely:

For every $a \in A$, there exists $a \ b \in B$ such that $(a, b) \in f$.

It is *deterministic*: every element of its domain occurs at most once in its graph. More precisely:

If $(a, b_1) \in f$ and $(a, b_2) \in f$, then $b_1 = b_2$.

We have seen how to define relations inductively. E.g.... Let *Consts* be the smallest two-place relation closed under the following rules:

 $(true, {true}) \in Consts$ $(false, {false}) \in Consts$ $(0, \{0\}) \in Consts$ $(t_1, C) \in Consts$ (succ t_1, C) \in Consts $(t_1, C) \in Consts$ (pred t_1, C) \in Consts $(t_1, C) \in Consts$ (iszero $t_1, C) \in Consts$ $(t_1, C_1) \in Consts$ $(t_2, C_2) \in Consts$ $(t_3, C_3) \in Consts$ (if t_1 then t_2 else t_3 , $C_1 \cup C_2 \cup C_3$) \in Consts

This definition certainly defines a *relation* (i.e., the smallest one with a certain closure property).

Q: How can we be sure that this relation is a function?

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Q: How can we be sure that this relation is a *function*?

A: Prove it!

Theorem:

The relation *Consts* defined by the inference rules a couple of slides ago is total and deterministic.

I.e., for each term t there is exactly one set of terms C such that $(t, C) \in Consts$.

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Proof: By induction on t.

To apply the induction principle for terms, we must show, for an arbitrary term t, that if

for each immediate subterm s of t, there is exactly one set of terms C_s such that $(s, C_s) \in Consts$

then

there is exactly one set of terms C such that $(t, C) \in Consts$.

Proceed by cases on the form of t.

If t is 0, true, or false, then we can immediately see from the definition of *Consts* that there is exactly one set of terms C (namely {t}) such that (t, C) ∈ *Consts*. Proceed by cases on the form of t.

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 (t₁, C₁) ∈ Consts. But then it is clear from the definition of
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 Consts that there is exactly one set C (namely C₁) such that
 (t, C) ∈ Consts.

Similarly when t is pred t_1 or iszero t_1 .

If t is if s₁ then s₂ else s₃, then the induction hypothesis tells us

• there is exactly one set of terms C_1 such that $(t_1, C_1) \in Consts$

- there is exactly one set of terms C_2 such that $(t_2, C_2) \in Consts$
- there is exactly one set of terms C_3 such that $(t_3, C_3) \in Consts$

But then it is clear from the definition of *Consts* that there is exactly one set *C* (namely $C_1 \cup C_2 \cup C_3$) such that $(t, C) \in Consts$.

How about the bad definition?

 $(true, {true}) \in BadConsts$ $(false, {false}) \in BadConsts$ $(0, \{0\}) \in BadConsts$ $(0, \{\}) \in BadConsts$ $(t_1, C) \in BadConsts$ (succ t_1, C) \in BadConsts $(t_1, C) \in BadConsts$ (pred t_1, C) \in BadConsts (iszero (iszero t_1), C) \in BadConsts (iszero t_1, C) \in BadConsts

Just for fun, let's calculate some cases of this relation...

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- For what values of C do we have
 - (if false then 0 else $0, C) \in BadConsts$?

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- For what values of C do we have (iszero 0, C) ∈ BadConsts?

Another Inductive Definition

Theorem: The number of distinct constants in a term is at most the size of the term. I.e., $|Consts(t)| \leq size(t)$.

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Assuming the desired property for immediate subterms of t, we must prove it for t itself.

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There are "three" cases to consider:

Case: t is a constant

Immediate: $|Consts(t)| = |\{t\}| = 1 = size(t)$.

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There are "three" cases to consider:

Case: t is a constant

Immediate: $|Consts(t)| = |\{t\}| = 1 = size(t)$.

Case: $t = succ t_1$, pred t_1 , or iszero t_1

By the induction hypothesis, $|Consts(t_1)| \le size(t_1)$. We now calculate as follows:

 $|Consts(t)| = |Consts(t_1)| \le size(t_1) < size(t_1) + 1 = size(t).$

Case: $t = if t_1 then t_2 else t_3$

By the induction hypothesis, $|Consts(t_1)| \le size(t_1)$, $|Consts(t_2)| \le size(t_2)$, and $|Consts(t_3)| \le size(t_3)$. We now calculate as follows:

$$\begin{split} |\textit{Consts}(\texttt{t})| &= |\textit{Consts}(\texttt{t}_1) \cup \textit{Consts}(\texttt{t}_2) \cup \textit{Consts}(\texttt{t}_3)| \\ &\leq |\textit{Consts}(\texttt{t}_1)| + |\textit{Consts}(\texttt{t}_2)| + |\textit{Consts}(\texttt{t}_3)| \\ &\leq \textit{size}(\texttt{t}_1) + \textit{size}(\texttt{t}_2) + \textit{size}(\texttt{t}_3) \\ &< \textit{size}(\texttt{t}). \end{split}$$

Operational Semantics

Abstract Machines

An abstract machine consists of:

a set of states

 \blacktriangleright a transition relation on states, written \longrightarrow

We read "t \longrightarrow t'" as "t evaluates to t' in one step".

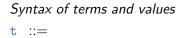
A state records *all* the information in the machine at a given moment. For example, an abstract-machine-style description of a conventional microprocessor would include the program counter, the contents of the registers, the contents of main memory, and the machine code program being executed.

Abstract Machines

For the very simple languages we are considering at the moment, however, the term being evaluated is the whole state of the abstract machine.

Nb. Often, the transition relation is actually a partial function: i.e., from a given state, there is at most one possible next state. But in general there may be many.

Operational semantics for Booleans



```
true
false
if t then t else t
```

v ::= true false terms constant true constant false conditional

values true value false value

Evaluation relation for Booleans

The evaluation relation $t \longrightarrow t'$ is the smallest relation closed under the following rules:

if true then t_2 else $t_3 \longrightarrow t_2$ (E-IFTRUE) if false then t_2 else $t_3 \longrightarrow t_3$ (E-IFFALSE) $\frac{t_1 \longrightarrow t'_1}{\text{if } t_1 \text{ then } t_2 \text{ else } t_3 \longrightarrow \text{if } t'_1 \text{ then } t_2 \text{ else } t_3} \text{(E-IF)}$



Computation rules:

if true then t_2 else $t_3 \longrightarrow t_2$ (E-IFTRUE) if false then t_2 else $t_3 \longrightarrow t_3$ (E-IFFALSE)

Congruence rule:

$$\frac{\texttt{t}_1 \longrightarrow \texttt{t}_1'}{\texttt{if } \texttt{t}_1 \texttt{ then } \texttt{t}_2 \texttt{ else } \texttt{t}_3 \longrightarrow \texttt{if } \texttt{t}_1' \texttt{ then } \texttt{t}_2 \texttt{ else } \texttt{t}_3} (\text{E-IF})$$

Computation rules perform "real" computation steps. Congruence rules determine *where* computation rules can be applied next.

Evaluation, more explicitly

 \longrightarrow is the smallest two-place relation closed under the following rules:

The notation $t \longrightarrow t'$ is short-hand for $(t, t') \in \longrightarrow$.

Derivations

We can record the "justification" for a particular pair of terms that are in the evaluation relation in the form of a tree.

(on the board)

Terminology:

- These trees are called *derivation trees* (or just *derivations*).
- The final statement in a derivation is its conclusion.
- We say that the derivation is a witness for its conclusion (or a proof of its conclusion) it records all the reasoning steps that justify the conclusion.

Observation

Lemma: Suppose we are given a derivation tree ${\cal D}$ witnessing the pair $(t,\,t')$ in the evaluation relation. Then either

- 1. the final rule used in \mathcal{D} is E-IFTRUE and we have $t = if true then t_2 else t_3 and t' = t_2$, for some t_2 and t_3 , or
- 2. the final rule used in \mathcal{D} is E-IFFALSE and we have $t = \text{if false then } t_2 \text{ else } t_3 \text{ and } t' = t_3$, for some t_2 and t_3 , or
- 3. the final rule used in \mathcal{D} is E-IF and we have $t = if t_1 then t_2 else t_3 and$ $t' = if t'_1 then t_2 else t_3$, for some t_1, t'_1, t_2 , and t_3 ; moreover, the immediate subderivation of \mathcal{D} witnesses $(t_1, t'_1) \in \longrightarrow$.

Induction on Derivations

We can now write proofs about evaluation "by induction on derivation trees."

Given an arbitrary derivation \mathcal{D} with conclusion $t \longrightarrow t'$, we assume the desired result for its immediate sub-derivation (if any) and proceed by a case analysis (using the previous lemma) of the final evaluation rule used in constructing the derivation tree.

E.g....

Induction on Derivations — Example

Theorem: If $t \to t'$, i.e., if $(t, t') \in \rightarrow$, then size(t) > size(t'). **Proof:** By induction on a derivation \mathcal{D} of $t \to t'$.

- 1. Suppose the final rule used in \mathcal{D} is E-IFTRUE, with $t = if true then t_2 else t_3 and t' = t_2$. Then the result is immediate from the definition of *size*.
- 2. Suppose the final rule used in \mathcal{D} is E-IFFALSE, with t = if false then t_2 else t_3 and $t' = t_3$. Then the result is again immediate from the definition of *size*.
- 3. Suppose the final rule used in \mathcal{D} is E-IF, with $t = if t_1 then t_2 else t_3$ and $t' = if t'_1 then t_2 else t_3$, where $(t_1, t'_1) \in \longrightarrow$ is witnessed by a derivation \mathcal{D}_1 . By the induction hypothesis, $size(t_1) > size(t'_1)$. But then, by the definition of size, we have size(t) > size(t').

Normal forms

A normal form is a term that cannot be evaluated any further — i.e., a term t is a normal form (or "is in normal form") if there is no t' such that $t \rightarrow t'$.

A normal form is a state where the abstract machine is halted — *i.e., it can be regarded as a "result" of evaluation.*

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A normal form is a state where the abstract machine is halted — *i.e., it can be regarded as a "result" of evaluation.*

Recall that we intended the set of *values* (the boolean constants true and false) to be exactly the possible "results of evaluation." Did we get this definition right?

Theorem: A term t is a value iff it is in normal form. **Proof:**

The \implies direction is immediate from the definition of the evaluation relation.

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For the \leftarrow direction, it is convenient to prove the contrapositive:

If t is not a value, then it is not a normal form.

Theorem: A term t is a value iff it is in normal form. **Proof:**

The \implies direction is immediate from the definition of the evaluation relation.

For the \Leftarrow direction, it is convenient to prove the contrapositive: If t is *not* a value, then it is *not* a normal form. The argument goes by induction on t.

Note, first, that t must have the form if t_1 then t_2 else t_3 (otherwise it would be a value). If t_1 is true or false, then rule E-IFTRUE or E-IFFALSE applies to t, and we are done. Otherwise, t_1 is not a value and so, by the induction hypothesis, there is some t'_1 such that $t_1 \longrightarrow t'_1$. But then rule E-IF yields

if
$$t_1$$
 then t_2 else $t_3 \longrightarrow$ if t_1' then t_2 else t_3

i.e., t is not in normal form.

Numbers

New syntactic forms

t ::= ... 0 succ t pred t iszero t

v ::= ... nv nv ::= 0

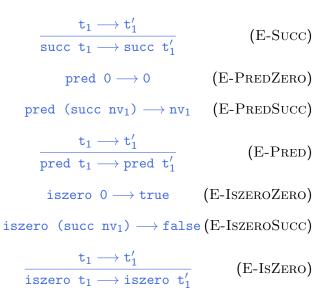
succ nv

terms constant zero successor predecessor zero test

values numeric value

numeric values zero value successor value New evaluation rules





Our observation a few slides ago that all values are in normal form still holds for the extended language.

Is the converse true? I.e., is every normal form a value?

Values are normal forms, but we have stuck terms

Our observation a few slides ago that all values are in normal form still holds for the extended language.

Is the converse true? I.e., is every normal form a value? No: some terms are *stuck*.

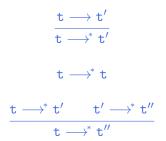
Formally, a stuck term is one that is a normal form but not a value. What are some examples?

Stuck terms model run-time errors.

Multi-step evaluation.

The *multi-step evaluation* relation, \rightarrow^* , is the reflexive, transitive closure of single-step evaluation.

I.e., it is the smallest relation closed under the following rules:



Termination of evaluation

Theorem: For every t there is some normal form t' such that $t \longrightarrow^* t'$. **Proof:**

Termination of evaluation

Theorem: For every t there is some normal form t' such that $t \longrightarrow^* t'$.

Proof:

First, recall that single-step evaluation strictly reduces the size of the term:

if $t \longrightarrow t'$, then size(t) > size(t')

Now, assume (for a contradiction) that $t_0, t_1, t_2, t_3, t_4, \ldots$

is an infinite-length sequence such that $t_0 \longrightarrow t_1 \longrightarrow t_2 \longrightarrow t_3 \longrightarrow t_4 \longrightarrow \cdots$.

Then $size(t_0) > size(t_1) > size(t_2) > size(t_3) > \dots$

But such a sequence cannot exist — contradiction!

Termination Proofs

Most termination proofs have the same basic form:

Theorem: The relation $R \subseteq X \times X$ is terminating — i.e., there are no infinite sequences x_0 , x_1 , x_2 , etc. such that $(x_i, x_{i+1}) \in R$ for each *i*.

- Proof:
 - 1. Choose
 - ► a well-founded set (W, <) i.e., a set W with a partial order < such that there are no infinite descending chains w₀ > w₁ > w₂ > ... in W
 - a function f from X to W
 - 2. Show f(x) > f(y) for all $(x, y) \in R$
 - Conclude that there are no infinite sequences x₀, x₁, x₂, etc. such that (x_i, x_{i+1}) ∈ R for each i, since, if there were, we could construct an infinite descending chain in W.