Theory of Types and Programming Languages Fall 2022

Week 7

Plan

PREVIOUSLY: unit, sequencing, let, pairs, tuples

TODAY:

1. options, variants
2. recursion
3. state

Records

$\mathtt{t}$	$::=$	...	*terms*
		$\{\mathtt{l}_i\mathtt{=}\mathtt{t}_i\ ^{i\in 1..n}\}$	*record*
		$\mathtt{t.l}$	*projection*
$\mathtt{v}$	$::=$	...	*values*
		$\{\mathtt{l}_i\mathtt{=}\mathtt{v}_i\ ^{i\in 1..n}\}$	*record value*
$\mathtt{T}$	$::=$	...	*types*
		$\{\mathtt{l}_i\mathtt{:}\mathtt{T}_i\ ^{i\in 1..n}\}$	*type of records*

Evaluation rules for records

$$\{\mathtt{l}_i\mathtt{=v}_i\ ^{i\in 1..n}\}\mathtt{.l}_j \longrightarrow \mathtt{v}_j \qquad \text{(E-ProjRcd)}$$

$$\frac{\mathtt{t}_1 \longrightarrow \mathtt{t}_1'}{\mathtt{t}_1\mathtt{.l} \longrightarrow \mathtt{t}_1'\mathtt{.l}} \qquad \text{(E-Proj)}$$

$$\frac{\mathtt{t}_j \longrightarrow \mathtt{t}_j'}{\begin{array}{c}\{\mathtt{l}_i\mathtt{=v}_i\ ^{i\in 1..j-1}\mathtt{,l}_j\mathtt{=t}_j\mathtt{,l}_k\mathtt{=t}_k\ ^{k\in j+1..n}\}\\ \longrightarrow \{\mathtt{l}_i\mathtt{=v}_i\ ^{i\in 1..j-1}\mathtt{,l}_j\mathtt{=t}_j'\mathtt{,l}_k\mathtt{=t}_k\ ^{k\in j+1..n}\}\end{array}} \qquad \text{(E-Rcd)}$$

Typing rules for records

$$\frac{\text{for each } i \qquad \Gamma \vdash \mathtt{t}_i : \mathtt{T}_i}{\Gamma \vdash \{\mathtt{l}_i\mathtt{=}\mathtt{t}_i\ ^{i\in 1..n}\} : \{\mathtt{l}_i\mathtt{:}\mathtt{T}_i\ ^{i\in 1..n}\}} \qquad \text{(T-Rcd)}$$

$$\frac{\Gamma \vdash \mathtt{t}_1 : \{\mathtt{l}_i\mathtt{:}\mathtt{T}_i\ ^{i\in 1..n}\}}{\Gamma \vdash \mathtt{t}_1\mathtt{.l}_j : \mathtt{T}_j} \qquad \text{(T-Proj)}$$

Sums and variants

Sums – motivating example

```
PhysicalAddr = {firstlast:String, addr:String}
VirtualAddr  = {name:String, email:String}
Addr         = PhysicalAddr + VirtualAddr
inl  :  "PhysicalAddr → PhysicalAddr+VirtualAddr"
inr  :  "VirtualAddr → PhysicalAddr+VirtualAddr"
```

```
getName = λa:Addr.
  case a of
    inl x ⇒ x.firstlast
  | inr y ⇒ y.name;
```

New syntactic forms

`t`	::=	...	*terms*
		`inl t`	*tagging (left)*
		`inr t`	*tagging (right)*
		`case t of inl x⇒t	inr x⇒t`	*case*
`v`	::=	...	*values*
		`inl v`	*tagged value (left)*
		`inr v`	*tagged value (right)*
`T`	::=	...	*types*
		`T+T`	*sum type*

T_1+T_2 is a *disjoint union* of T_1 and T_2 (the tags `inl` and `inr` ensure disjointness)

New evaluation rules

$$\boxed{\mathtt{t} \longrightarrow \mathtt{t}'}$$

$$\begin{array}{l}\mathtt{case\ (inl\ v_0)} \\ \mathtt{of\ inl\ x_1{\Rightarrow}t_1\ |\ inr\ x_2{\Rightarrow}t_2}\end{array} \longrightarrow [\mathtt{x_1} \mapsto \mathtt{v_0}]\mathtt{t_1} \qquad \text{(E-CaseInl)}$$

$$\begin{array}{l}\mathtt{case\ (inr\ v_0)} \\ \mathtt{of\ inl\ x_1{\Rightarrow}t_1\ |\ inr\ x_2{\Rightarrow}t_2}\end{array} \longrightarrow [\mathtt{x_2} \mapsto \mathtt{v_0}]\mathtt{t_2} \qquad \text{(E-CaseInr)}$$

$$\frac{\mathtt{t_0} \longrightarrow \mathtt{t_0'}}{\begin{array}{l}\mathtt{case\ t_0\ of\ inl\ x_1{\Rightarrow}t_1\ |\ inr\ x_2{\Rightarrow}t_2} \\ \longrightarrow \mathtt{case\ t_0'\ of\ inl\ x_1{\Rightarrow}t_1\ |\ inr\ x_2{\Rightarrow}t_2}\end{array}} \qquad \text{(E-Case)}$$

$$\frac{\mathtt{t_1} \longrightarrow \mathtt{t_1'}}{\mathtt{inl\ t_1} \longrightarrow \mathtt{inl\ t_1'}} \qquad \text{(E-Inl)}$$

$$\frac{\mathtt{t_1} \longrightarrow \mathtt{t_1'}}{\mathtt{inr\ t_1} \longrightarrow \mathtt{inr\ t_1'}} \qquad \text{(E-Inr)}$$

New typing rules

$$\boxed{\Gamma \vdash \mathtt{t} : \mathtt{T}}$$

$$\frac{\Gamma \vdash \mathtt{t}_1 : \mathtt{T}_1}{\Gamma \vdash \mathtt{inl}\ \mathtt{t}_1 : \mathtt{T}_1\mathtt{+}\mathtt{T}_2} \qquad (\text{T-Inl})$$

$$\frac{\Gamma \vdash \mathtt{t}_1 : \mathtt{T}_2}{\Gamma \vdash \mathtt{inr}\ \mathtt{t}_1 : \mathtt{T}_1\mathtt{+}\mathtt{T}_2} \qquad (\text{T-Inr})$$

$$\frac{\begin{array}{c}\Gamma \vdash \mathtt{t}_0 : \mathtt{T}_1\mathtt{+}\mathtt{T}_2 \\ \Gamma, \mathtt{x}_1{:}\mathtt{T}_1 \vdash \mathtt{t}_1 : \mathtt{T} \qquad \Gamma, \mathtt{x}_2{:}\mathtt{T}_2 \vdash \mathtt{t}_2 : \mathtt{T}\end{array}}{\Gamma \vdash \mathtt{case}\ \mathtt{t}_0\ \mathtt{of}\ \mathtt{inl}\ \mathtt{x}_1 \Rightarrow \mathtt{t}_1\ \mathtt{|}\ \mathtt{inr}\ \mathtt{x}_2 \Rightarrow \mathtt{t}_2 : \mathtt{T}} \qquad (\text{T-Case})$$

Sums and Uniqueness of Types

Problem:

If `t` has type `T`, then `inl t` has type `T+U` for every *`U`.*

I.e., we've lost uniqueness of types.

Possible solutions:

- "Infer" `U` as needed during typechecking
- Pre-declare sum types and associate their constructors with fixed types (e.g., `type U` $=$ `inl`$_{\texttt{U}}$ `Nat` $+$ `inr`$_{\texttt{U}}$ `Bool`)
- Annotate each `inl` and `inr` with the intended sum type

For simplicity, let's choose the third one.

New syntactic forms

`t`	::=	...	*terms*
		`inl t as T`	*tagging (left)*
		`inr t as T`	*tagging (right)*
`v`	::=	...	*values*
		`inl v as T`	*tagged value (left)*
		`inr v as T`	*tagged value (right)*

Note that `as T` here is not the ascription operator that we saw before — i.e., not a separate syntactic form: in essence, there is an ascription "built into" every use of `inl` or `inr`.

New typing rules

$$\boxed{\Gamma \vdash \mathtt{t} : \mathtt{T}}$$

$$\frac{\Gamma \vdash \mathtt{t}_1 : \mathtt{T}_1}{\Gamma \vdash \mathtt{inl}\ \mathtt{t}_1\ \mathtt{as}\ \mathtt{T}_1\mathtt{+}\mathtt{T}_2 : \mathtt{T}_1\mathtt{+}\mathtt{T}_2} \qquad (\textsc{T-Inl})$$

$$\frac{\Gamma \vdash \mathtt{t}_1 : \mathtt{T}_2}{\Gamma \vdash \mathtt{inr}\ \mathtt{t}_1\ \mathtt{as}\ \mathtt{T}_1\mathtt{+}\mathtt{T}_2 : \mathtt{T}_1\mathtt{+}\mathtt{T}_2} \qquad (\textsc{T-Inr})$$

Evaluation rules ignore annotations:

$\boxed{\mathtt{t} \longrightarrow \mathtt{t}'}$

$$\begin{array}{l}\mathtt{case\ (inl\ v_0\ as\ T_0)} \\ \mathtt{of\ inl\ x_1 \Rightarrow t_1\ \mid\ inr\ x_2 \Rightarrow t_2} \\ \qquad \longrightarrow [\mathtt{x_1} \mapsto \mathtt{v_0}]\mathtt{t_1}\end{array} \tag{E-CaseInl}$$

$$\begin{array}{l}\mathtt{case\ (inr\ v_0\ as\ T_0)} \\ \mathtt{of\ inl\ x_1 \Rightarrow t_1\ \mid\ inr\ x_2 \Rightarrow t_2} \\ \qquad \longrightarrow [\mathtt{x_2} \mapsto \mathtt{v_0}]\mathtt{t_2}\end{array} \tag{E-CaseInr}$$

$$\frac{\mathtt{t_1} \longrightarrow \mathtt{t_1'}}{\mathtt{inl\ t_1\ as\ T_2} \longrightarrow \mathtt{inl\ t_1'\ as\ T_2}} \tag{E-Inl}$$

$$\frac{\mathtt{t_1} \longrightarrow \mathtt{t_1'}}{\mathtt{inr\ t_1\ as\ T_2} \longrightarrow \mathtt{inr\ t_1'\ as\ T_2}} \tag{E-Inr}$$

Variants

Just as we generalized binary products to labeled records, we can generalize binary sums to labeled *variants*.

New syntactic forms

$\mathtt{t}$	::=	...	*terms*
		$\mathtt{<l=t>\ as\ T}$	*tagging*
		$\mathtt{case\ t\ of\ <l}_i\mathtt{=x}_i\mathtt{>} \Rightarrow \mathtt{t}_i \ ^{i \in 1..n}$	*case*
$\mathtt{T}$	::=	...	*types*
		$\mathtt{<l}_i\mathtt{:T}_i \ ^{i \in 1..n}\mathtt{>}$	*type of variants*

New evaluation rules

$\boxed{\texttt{t} \longrightarrow \texttt{t}'}$

$$\begin{array}{l}\texttt{case (<l}_j\texttt{=v}_j\texttt{> as T) of <l}_i\texttt{=x}_i\texttt{>}\Rightarrow\texttt{t}_i\ {}^{i\in 1..n} \\ \qquad\longrightarrow [\texttt{x}_j \mapsto \texttt{v}_j]\texttt{t}_j\end{array} \qquad \text{(E-CaseVariant)}$$

$$\frac{\texttt{t}_0 \longrightarrow \texttt{t}_0'}{\begin{array}{l}\texttt{case t}_0\texttt{ of <l}_i\texttt{=x}_i\texttt{>}\Rightarrow\texttt{t}_i\ {}^{i\in 1..n} \\ \longrightarrow \texttt{case t}_0'\texttt{ of <l}_i\texttt{=x}_i\texttt{>}\Rightarrow\texttt{t}_i\ {}^{i\in 1..n}\end{array}} \qquad \text{(E-Case)}$$

$$\frac{\texttt{t}_i \longrightarrow \texttt{t}_i'}{\texttt{<l}_i\texttt{=t}_i\texttt{> as T} \longrightarrow \texttt{<l}_i\texttt{=t}_i'\texttt{> as T}} \qquad \text{(E-Variant)}$$

New typing rules

$$\boxed{\Gamma \vdash \mathtt{t} : \mathtt{T}}$$

$$\frac{\Gamma \vdash \mathtt{t}_j : \mathtt{T}_j}{\Gamma \vdash \mathtt{<l}_j\mathtt{=t}_j\mathtt{>}\ \mathtt{as}\ \mathtt{<l}_i\mathtt{:T}_i\ ^{i\in 1..n}\mathtt{>} : \mathtt{<l}_i\mathtt{:T}_i\ ^{i\in 1..n}\mathtt{>}} \quad \text{(T-Variant)}$$

$$\frac{\begin{array}{c}\Gamma \vdash \mathtt{t}_0 : \mathtt{<l}_i\mathtt{:T}_i\ ^{i\in 1..n}\mathtt{>} \\ \text{for each } i \quad \Gamma, \mathtt{x}_i\mathtt{:T}_i \vdash \mathtt{t}_i : \mathtt{T}\end{array}}{\Gamma \vdash \mathtt{case}\ \mathtt{t}_0\ \mathtt{of}\ \mathtt{<l}_i\mathtt{=x}_i\mathtt{>}\Rightarrow\mathtt{t}_i\ ^{i\in 1..n} : \mathtt{T}} \quad \text{(T-Case)}$$

Example

```
Addr = <physical:PhysicalAddr, virtual:VirtualAddr>;

a = <physical=pa> as Addr;

getName = λa:Addr.
  case a of
    <physical=x> ⇒ x.firstlast
  | <virtual=y> ⇒ y.name;
```

Options

```
OptionalNat = <none:Unit, some:Nat>;

Table = Nat→OptionalNat;

emptyTable = λn:Nat. <none=unit> as OptionalNat;

extendTable =
  λt:Table. λm:Nat. λv:Nat.
    λn:Nat.
      if equal n m then <some=v> as OptionalNat
      else t n;

x = case t(5) of
      <none=u> ⇒ 999
    | <some=v> ⇒ v;
```

Enumerations

```
Weekday = <monday:Unit, tuesday:Unit, wednesday:Unit,
           thursday:Unit, friday:Unit>;

nextBusinessDay = λw:Weekday.
  case w of <monday=x>    ⇒ <tuesday=unit> as Weekday
          | <tuesday=x>   ⇒ <wednesday=unit> as Weekday
          | <wednesday=x> ⇒ <thursday=unit> as Weekday
          | <thursday=x>  ⇒ <friday=unit> as Weekday
          | <friday=x>    ⇒ <monday=unit> as Weekday;
```

Recursion

Recursion in $\lambda_{\rightarrow}$

- In $\lambda_{\rightarrow}$, all programs terminate. (Cf. Chapter 12.)
- Hence, untyped terms like `omega` and `fix` are not typable.
- But we can *extend* the system with a (typed) fixed-point operator...

Example

```
ff = λie:Nat→Bool.
       λx:Nat.
         if iszero x then true
         else if iszero (pred x) then false
         else ie (pred (pred x));

iseven = fix ff;

iseven 7;
```

New syntactic forms

$$\begin{array}{lll} \mathtt{t} & ::= & \ldots \\ & & \mathtt{fix}\ \mathtt{t} \end{array} \qquad \begin{array}{l} \textit{terms} \\ \quad \textit{fixed point of } t \end{array}$$

New evaluation rules $\boxed{\mathtt{t} \longrightarrow \mathtt{t}'}$

$$\begin{array}{c} \mathtt{fix}\ (\lambda \mathtt{x}{:}\mathtt{T}_1.\mathtt{t}_2) \\ \longrightarrow [\mathtt{x} \mapsto (\mathtt{fix}\ (\lambda \mathtt{x}{:}\mathtt{T}_1.\mathtt{t}_2))]\mathtt{t}_2 \end{array} \qquad \text{(E-FixBeta)}$$

$$\frac{\mathtt{t}_1 \longrightarrow \mathtt{t}_1'}{\mathtt{fix}\ \mathtt{t}_1 \longrightarrow \mathtt{fix}\ \mathtt{t}_1'} \qquad \text{(E-Fix)}$$

New typing rules

$$\boxed{\Gamma \vdash \mathtt{t} : \mathtt{T}}$$

$$\frac{\Gamma \vdash \mathtt{t}_1 : \mathtt{T}_1 \to \mathtt{T}_1}{\Gamma \vdash \mathtt{fix}\ \mathtt{t}_1 : \mathtt{T}_1} \qquad \text{(T-Fix)}$$

A more convenient form

$\texttt{letrec x:T}_1\texttt{=t}_1\texttt{ in t}_2 \overset{\text{def}}{=} \texttt{let x = fix (}\lambda\texttt{x:T}_1\texttt{.t}_1\texttt{) in t}_2$

```
letrec iseven : Nat→Bool =
  λx:Nat.
    if iszero x then true
    else if iszero (pred x) then false
    else iseven (pred (pred x))
in
  iseven 7;
```

References

Mutability

- In most programming languages, *variables* are mutable — i.e., a variable provides both
  - a name that refers to a previously calculated value, and
  - the possibility of *overwriting* this value with another (which will be referred to by the same name)

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- In some languages (e.g., OCaml), these features are separate:
  - variables are only for naming — the binding between a variable and its value is immutable
  - introduce a new class of *mutable values* (called *reference cells* or *references*)
  - at any given moment, a reference holds a value (and can be *dereferenced* to obtain this value)
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  - a new value may be *assigned* to a reference

We choose OCaml's style, which is easier to work with formally.

So a variable of type `T` in most languages (except OCaml) will correspond to a `Ref T` (actually, a `Ref(Option T)`) here.

Basic Examples

```
r = ref 5

!r

r := 7

(r:=succ(!r); !r)

(r:=succ(!r); r:=succ(!r); r:=succ(!r);
 r:=succ(!r); !r)
```

Basic Examples

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r = ref 5
```

```
!r
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```
r := 7
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```
(r:=succ(!r); !r)
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```
(r:=succ(!r); r:=succ(!r); r:=succ(!r);
 r:=succ(!r); !r)
```

i.e.,

```
((((r:=succ(!r); r:=succ(!r)); r:=succ(!r));
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Mutable Aliasing

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Aliasing becomes more "interesting" when mutability is involved.

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to a cell holding a value of type `T`.

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If this value is "copied" by assigning it to another variable, the cell pointed to is not copied.

So we can change `r` by assigning to `s`:

```
(s := 6; !r)
```

Aliasing all around us

Reference cells are not the only language feature that introduces the possibility of aliasing.

- object references
- explicit pointers in C
- arrays
- communication channels
- I/O devices (disks, etc.)

The difficulties of aliasing

The possibility of aliasing invalidates all sorts of useful forms of reasoning about programs, both by programmers...

The function λ`r:Ref Nat.` λ`s:Ref Nat. (r:=2; s:=3; !r)`
always returns `2` unless `r` *and* `s` *are aliases.*

...and by compilers:

Code motion out of loops, common subexpression elimination, allocation of variables to registers, and detection of uninitialized variables all depend upon the compiler knowing which objects a load or a store operation could reference.

High-performance compilers spend significant energy on *alias analysis* to try to establish when different variables cannot possibly refer to the same storage.

The difficulties of side effects

The order of operations now matters.

```
f (r := 1) (r := 2)
```

Benefits of aliasing

The problems of *mutable* aliasing have led some language designers to restrict it (e.g., Rust) or to simply remove mutability (e.g., Haskell)

But there are good reasons why most languages do provide constructs involving mutable aliasing:

- efficiency (e.g., arrays)
- "action at a distance" (e.g., symbol tables, graphs)
- dependency-driven data flow (e.g., in GUI's)
- shared resources (e.g., locks) in concurrent systems
- etc.

Example

```
c = ref 0
incc = λx:Unit. (c := succ (!c); !c)
decc = λx:Unit. (c := pred (!c); !c)
incc unit
decc unit
o = {i = incc, d = decc}
```

```
let newcounter =
  λ_:Unit.
    let c = ref 0 in
    let incc = λx:Unit. (c := succ (!c); !c) in
    let decc = λx:Unit. (c := pred (!c); !c) in
    let o = {i = incc, d = decc} in
    o
```

Syntax

$\texttt{t}$	::=		*terms*
		$\texttt{unit}$	*unit constant*
		$\texttt{x}$	*variable*
		$\lambda\texttt{x:T.t}$	*abstraction*
		$\texttt{t t}$	*application*
		$\texttt{ref t}$	*reference creation*
		$\texttt{!t}$	*dereference*
		$\texttt{t:=t}$	*assignment*

... plus other familiar types, in examples.

Typing Rules

$$\frac{\Gamma \vdash \mathtt{t_1} : \mathtt{T_1}}{\Gamma \vdash \mathtt{ref\ t_1} : \mathtt{Ref\ T_1}} \qquad \text{(T-Ref)}$$

$$\frac{\Gamma \vdash \mathtt{t_1} : \mathtt{Ref\ T_1}}{\Gamma \vdash \mathtt{!t_1} : \mathtt{T_1}} \qquad \text{(T-Deref)}$$

$$\frac{\Gamma \vdash \mathtt{t_1} : \mathtt{Ref\ T_1} \qquad \Gamma \vdash \mathtt{t_2} : \mathtt{T_1}}{\Gamma \vdash \mathtt{t_1 := t_2} : \mathtt{Unit}} \qquad \text{(T-Assign)}$$

Final example

```
NatArray = Ref (Nat→Nat);

newarray = λ_:Unit. ref (λn:Nat.0);
         : Unit → NatArray

lookup = λa:NatArray. λn:Nat. (!a) n;
       : NatArray → Nat → Nat

update = λa:NatArray. λm:Nat. λv:Nat.
           let oldf = !a in
           a := (λn:Nat. if equal m n then v else oldf n);
       : NatArray → Nat → Nat → Unit
```

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Crucial observation: evaluating `ref 0` must *do* something.
Otherwise,

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r = ref 0
s = ref 0
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Specifically, evaluating `ref 0` should *allocate some storage* and yield a *reference* (or *pointer*) to that storage.
So what is a reference?

The Store

A reference names a *location* in the *store* (also known as the *heap* or just the *memory*).

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The Store

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What is the store?

- *Concretely:* An array of 8-bit bytes, indexed by 32-bit integers.
- *More abstractly:* an array of *values*
- *Even more abstractly:* a partial function from *locations* to *values*.

Locations

Syntax of values:

$\mathtt{v}$ ::=		*values*
	$\mathtt{unit}$	*unit constant*
	$\lambda\mathtt{x:T.t}$	*abstraction value*
	l	*store location*

... and since all values are terms...

Syntax of Terms

`t`	::=		*terms*
		`unit`	*unit constant*
		`x`	*variable*
		λ`x:T.t`	*abstraction*
		`t t`	*application*
		`ref t`	*reference creation*
		`!t`	*dereference*
		`t:=t`	*assignment*
		l	store location

Aside

Does this mean we are going to allow programmers to write explicit locations in their programs??

No: This is just a modeling trick. We are enriching the "source language" to include some run-time structures, so that we can continue to formalize evaluation as a relation between source terms.

Aside: If we formalize evaluation in the big-step style, then we can add locations to the set of values (results of evaluation) without adding them to the set of terms.

Evaluation

The result of evaluating a term now depends on the store in which it is evaluated. Moreover, the result of evaluating a term is not just a value — we must also keep track of the changes that get made to the store.

I.e., the evaluation relation should now map a term and a store to a reduced term and a new store.

$$\mathtt{t} \,|\, \mu \longrightarrow \mathtt{t}' \,|\, \mu'$$

We use the metavariable μ to range over stores.

Evaluation

An assignment $\mathtt{t}_1\mathtt{:=}\mathtt{t}_2$ first evaluates $\mathtt{t}_1$ and $\mathtt{t}_2$ until they become values...

$$\frac{\mathtt{t}_1 \mid \mu \longrightarrow \mathtt{t}_1' \mid \mu'}{\mathtt{t}_1\mathtt{:=}\mathtt{t}_2 \mid \mu \longrightarrow \mathtt{t}_1'\mathtt{:=}\mathtt{t}_2 \mid \mu'} \qquad \text{(E-ASSIGN1)}$$

$$\frac{\mathtt{t}_2 \mid \mu \longrightarrow \mathtt{t}_2' \mid \mu'}{\mathtt{v}_1\mathtt{:=}\mathtt{t}_2 \mid \mu \longrightarrow \mathtt{v}_1\mathtt{:=}\mathtt{t}_2' \mid \mu'} \qquad \text{(E-ASSIGN2)}$$

... and then returns $\mathtt{unit}$ and updates the store:

$$l\mathtt{:=}\mathtt{v}_2 \mid \mu \longrightarrow \mathtt{unit} \mid [l \mapsto \mathtt{v}_2]\mu \qquad \text{(E-ASSIGN)}$$

Note: The $[l \mapsto \mathtt{v}_2]\mu$ notation is for heap update;

it does *not* denote substitution (confusingly)

A term of the form $\texttt{ref t}_1$ first evaluates inside $\texttt{t}_1$ until it becomes a value...

$$\frac{\texttt{t}_1 \mid \mu \longrightarrow \texttt{t}_1' \mid \mu'}{\texttt{ref t}_1 \mid \mu \longrightarrow \texttt{ref t}_1' \mid \mu'} \qquad \text{(E-Ref)}$$

... and then chooses (allocates) a fresh location l, augments the store with a binding from l to $\texttt{v}_1$, and returns l:

$$\frac{l \notin dom(\mu)}{\texttt{ref v}_1 \mid \mu \longrightarrow l \mid (\mu, l \mapsto \texttt{v}_1)} \qquad \text{(E-RefV)}$$

A term $\texttt{!t}_1$ first evaluates in $\texttt{t}_1$ until it becomes a value...

$$\frac{\texttt{t}_1 \mid \mu \longrightarrow \texttt{t}_1' \mid \mu'}{\texttt{!t}_1 \mid \mu \longrightarrow \texttt{!t}_1' \mid \mu'} \qquad \text{(E-Deref)}$$

... and then looks up this value (which must be a location, if the original term was well typed) and returns its contents in the current store:

$$\frac{\mu(l) = \texttt{v}}{\texttt{!}l \mid \mu \longrightarrow \texttt{v} \mid \mu} \qquad \text{(E-DerefLoc)}$$

Evaluation rules for function abstraction and application are augmented with stores, but don't do anything with them directly.

$$\frac{\mathtt{t}_1 | \, \mu \longrightarrow \mathtt{t}_1' | \, \mu'}{\mathtt{t}_1 \;\; \mathtt{t}_2 | \, \mu \longrightarrow \mathtt{t}_1' \;\; \mathtt{t}_2 | \, \mu'} \qquad \text{(E-APP1)}$$

$$\frac{\mathtt{t}_2 | \, \mu \longrightarrow \mathtt{t}_2' | \, \mu'}{\mathtt{v}_1 \;\; \mathtt{t}_2 | \, \mu \longrightarrow \mathtt{v}_1 \;\; \mathtt{t}_2' | \, \mu'} \qquad \text{(E-APP2)}$$

$$(\lambda\mathtt{x}{:}\mathtt{T}_{11}.\mathtt{t}_{12}) \;\; \mathtt{v}_2 | \, \mu \longrightarrow [\mathtt{x} \mapsto \mathtt{v}_2]\mathtt{t}_{12} | \, \mu \qquad \text{(E-APPABS)}$$

Aside: garbage collection

Note that we are not modeling garbage collection — the store just grows without bound.

Aside: pointer arithmetic

We can't do any!

Store Typings

Typing Locations

Q: What is the *type* of a *location*?

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A: It depends on the store!

E.g., in the store $(l_1 \mapsto \texttt{unit}, l_2 \mapsto \texttt{unit})$, the term $\texttt{!}\,l_2$ has type $\texttt{Unit}$.

But in the store $(l_1 \mapsto \texttt{unit}, l_2 \mapsto \lambda\texttt{x:Unit.x})$, the term $\texttt{!}\,l_2$ has type $\texttt{Unit}\to\texttt{Unit}$.

Typing Locations — first try

Roughly:

$$\frac{\Gamma \vdash \mu(l) : \mathtt{T_1}}{\Gamma \vdash l : \mathtt{Ref\ \ T_1}}$$

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More precisely:

$$\frac{\Gamma \mid \mu \vdash \mu(l) : \mathtt{T_1}}{\Gamma \mid \mu \vdash l : \mathtt{Ref\ T_1}}$$

I.e., typing is now a *four*-place relation (between contexts, *stores*, terms, and types).

Problem

However, this rule is not completely satisfactory. For one thing, it can make typing derivations very large!

E.g., if

$$
\begin{aligned}
(\mu = l_1 &\mapsto \lambda\texttt{x:Nat. 999},\\
l_2 &\mapsto \lambda\texttt{x:Nat. !}l_1\ \texttt{(!}l_1\ \texttt{x)},\\
l_3 &\mapsto \lambda\texttt{x:Nat. !}l_2\ \texttt{(!}l_2\ \texttt{x)},\\
l_4 &\mapsto \lambda\texttt{x:Nat. !}l_3\ \texttt{(!}l_3\ \texttt{x)},\\
l_5 &\mapsto \lambda\texttt{x:Nat. !}l_4\ \texttt{(!}l_4\ \texttt{x)}),
\end{aligned}
$$

then how big is the typing derivation for $\texttt{!}l_5$?

Problem!

But wait... it gets worse. Suppose

$$\begin{aligned}(\mu = l_1 &\mapsto \lambda\texttt{x:Nat. }\,!l_2\ \texttt{x},\\ l_2 &\mapsto \lambda\texttt{x:Nat. }\,!l_1\ \texttt{x}),\end{aligned}$$

Now how big is the typing derivation for $!l_2$?

Store Typings

Observation: The typing rules we have chosen for references guarantee that a given location in the store is *always* used to hold values of the *same* type.

These intended types can be collected into a *store typing* — a partial function from locations to types.

E.g., for

$$\begin{aligned}\mu = (l_1 &\mapsto \lambda\texttt{x:Nat. 999},\\ l_2 &\mapsto \lambda\texttt{x:Nat. !}l_1\texttt{ (!}l_1\texttt{ x)},\\ l_3 &\mapsto \lambda\texttt{x:Nat. !}l_2\texttt{ (!}l_2\texttt{ x)},\\ l_4 &\mapsto \lambda\texttt{x:Nat. !}l_3\texttt{ (!}l_3\texttt{ x)},\\ l_5 &\mapsto \lambda\texttt{x:Nat. !}l_4\texttt{ (!}l_4\texttt{ x)}),\end{aligned}$$

A reasonable store typing would be

$$\begin{aligned}\Sigma = (l_1 &\mapsto \texttt{Nat}\to\texttt{Nat},\\ l_2 &\mapsto \texttt{Nat}\to\texttt{Nat},\\ l_3 &\mapsto \texttt{Nat}\to\texttt{Nat},\\ l_4 &\mapsto \texttt{Nat}\to\texttt{Nat},\\ l_5 &\mapsto \texttt{Nat}\to\texttt{Nat})\end{aligned}$$

Now, suppose we are given a store typing Σ describing the store μ in which we intend to evaluate some term $\mathtt{t}$. Then we can use Σ to look up the types of locations in $\mathtt{t}$ instead of calculating them from the values in μ.

$$\frac{\Sigma(l) = \mathtt{T_1}}{\Gamma \mid \Sigma \vdash l : \mathtt{Ref\ T_1}} \qquad \text{(T-Loc)}$$

I.e., typing is now a four-place relation between between contexts, *store typings*, terms, and types.

Final typing rules

$$\frac{\Sigma(l) = \mathtt{T_1}}{\Gamma \mid \Sigma \vdash l : \mathtt{Ref\ T_1}} \qquad \text{(T-Loc)}$$

$$\frac{\Gamma \mid \Sigma \vdash \mathtt{t_1} : \mathtt{T_1}}{\Gamma \mid \Sigma \vdash \mathtt{ref\ t_1} : \mathtt{Ref\ T_1}} \qquad \text{(T-Ref)}$$

$$\frac{\Gamma \mid \Sigma \vdash \mathtt{t_1} : \mathtt{Ref\ T_{11}}}{\Gamma \mid \Sigma \vdash \mathtt{!t_1} : \mathtt{T_{11}}} \qquad \text{(T-Deref)}$$

$$\frac{\Gamma \mid \Sigma \vdash \mathtt{t_1} : \mathtt{Ref\ T_{11}} \qquad \Gamma \mid \Sigma \vdash \mathtt{t_2} : \mathtt{T_{11}}}{\Gamma \mid \Sigma \vdash \mathtt{t_1{:=}t_2} : \mathtt{Unit}} \qquad \text{(T-Assign)}$$

Q: Where do these store typings come from?

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A: When we first typecheck a program, there will be no explicit locations, so we can use an empty store typing.

So, when a new location is created during evaluation,

$$\frac{l \notin dom(\mu)}{\texttt{ref}\ \ \texttt{v}_1 \mid \mu \longrightarrow l \mid (\mu, l \mapsto \texttt{v}_1)} \qquad \text{(E-RefV)}$$

we can extend the "current store typing" with the type of $\texttt{v}_1$.

Safety

Preservation

First attempt: just add stores and store typings in the appropriate places.

Theorem (?): If $\Gamma \mid \Sigma \vdash \mathtt{t} : \mathtt{T}$ and $\mathtt{t} \mid \mu \longrightarrow \mathtt{t}' \mid \mu'$, then $\Gamma \mid \Sigma \vdash \mathtt{t}' : \mathtt{T}$.

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Why is this wrong?

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Why is this wrong?

Because Σ and μ here are not constrained to have anything to do with each other!

(Exercise: Construct an example that breaks this statement of preservation.)

Preservation

A store μ is said to be *well typed* with respect to a typing context Γ and a store typing Σ, written $\Gamma \mid \Sigma \vdash \mu$, if $dom(\mu) = dom(\Sigma)$ and $\Gamma \mid \Sigma \vdash \mu(l) : \Sigma(l)$ for every $l \in dom(\mu)$.

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Next attempt:

Theorem (?): If

$\Gamma \mid \Sigma \vdash \mathtt{t} : \mathtt{T}$

$\mathtt{t} \mid \mu \longrightarrow \mathtt{t}' \mid \mu'$

$\Gamma \mid \Sigma \vdash \mu$

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What's wrong now?

Preservation

A store μ is said to be *well typed* with respect to a typing context Γ and a store typing Σ, written $\Gamma \mid \Sigma \vdash \mu$, if $dom(\mu) = dom(\Sigma)$ and $\Gamma \mid \Sigma \vdash \mu(l) : \Sigma(l)$ for every $l \in dom(\mu)$.

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$$\Gamma \mid \Sigma \vdash \mathtt{t} : \mathtt{T}$$

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$$\Gamma \mid \Sigma \vdash \mu$$

then $\Gamma \mid \Sigma \vdash \mathtt{t}' : \mathtt{T}$. Still wrong!

Creation of a new reference cell...

$$\frac{l \notin dom(\mu)}{\mathtt{ref}\ \mathtt{v}_1 \mid \mu \longrightarrow l \mid (\mu, l \mapsto \mathtt{v}_1)} \qquad \text{(E-RefV)}$$

... breaks the correspondence between the store typing and the store.

Preservation (correct version)

Theorem: If

$$\Gamma \mid \Sigma \vdash \mathtt{t} : \mathtt{T}$$
$$\Gamma \mid \Sigma \vdash \mu$$
$$\mathtt{t} \mid \mu \longrightarrow \mathtt{t}' \mid \mu'$$

then, for some $\Sigma' \supseteq \Sigma$,

$$\Gamma \mid \Sigma' \vdash \mathtt{t}' : \mathtt{T}$$
$$\Gamma \mid \Sigma' \vdash \mu'.$$

Preservation (correct version)

Theorem: If

$$\Gamma \mid \Sigma \vdash \mathtt{t} : \mathtt{T}$$
$$\Gamma \mid \Sigma \vdash \mu$$
$$\mathtt{t} \mid \mu \longrightarrow \mathtt{t}' \mid \mu'$$

then, for some $\Sigma' \supseteq \Sigma$,

$$\Gamma \mid \Sigma' \vdash \mathtt{t}' : \mathtt{T}$$
$$\Gamma \mid \Sigma' \vdash \mu'.$$

Proof: Easy extension of the preservation proof for $\lambda_{\rightarrow}$.

Progress

Theorem: Suppose $\mathtt{t}$ is a closed, well-typed term (that is, $\emptyset \mid \Sigma \vdash \mathtt{t} : \mathtt{T}$ for some $\mathtt{T}$ and Σ). Then either $\mathtt{t}$ is a value or else, for any store μ such that $\emptyset \mid \Sigma \vdash \mu$, there is some term $\mathtt{t}'$ and store μ' with $\mathtt{t} \mid \mu \longrightarrow \mathtt{t}' \mid \mu'$.